Pergamon

Int. J. Solids Structures Vol. 33, No. 14, pp. 2039–2054, 1996

0020–7683/96 $15.00 + .00

0020–7683(95)00150–6

STRESS SINGULARITIES IN A BIMATERIAL JOINT WITH INHOMOGENEOUS TEMPERATURE DISTRIBUTION

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(*Received* 5 *December* 1994; *in revised form* 26 *June* 1995)

Abstract—The stress distribution near the free edge of the interface of a dissimilar materials joint with inhomogeneous temperature distribution can be described analytically by a two-terms relation. The stress intensity factor of the singular term is determined from finite element calculations. A general expression of the stress intensity factor for a polynomial temperature distribution is presented.

NOTATION

C_{Hij}	coefficients for stress intensity factor, see eqn (12)
C_{Lij}	coefficients for stress intensity factor, see eqn (8a)
C^*_{Lij}	coefficients for stress intensity factor, see eqn (10)
D^*_{Lij}	C^*_{Lij}/α^*_j
E	Young's modulus
H_i	height of joint
f_{ij}	angular functions of singular term
f_{ijo}	angular functions of the regular term
K_H	stress intensity factor (r related H)
K_L	stress intensity factor (r related L)
K_{Lh}	stress intensity factor for homogeneous change in temperature by one Kelvin (r related L)
L	half length of the interface
m_{ij}	coefficients in the expression of a polynomial temperature distribution
r, θ	polar coordinates
T	temperature
T_o	temperature of stress free state
ΔT	temperature difference $T - T_o$
α	thermal expansion coefficient
σ_{ij}	components of stress tensor
σ_o	constant (regular) stress term
ν	Poisson's ratio
ω	stress exponent

1. INTRODUCTION

In a joint of two dissimilar materials very high stresses develop near the free edge of the interface after a change in temperature. So far these stresses have been calculated mainly for a homogeneous change in temperature (see Kuo (1989), Suhir (1989), Suga *et al.* (1989), Blanchard and Ghoneim (1989), Kimura and Kawashima (1989), Eischen *et al.* (1990), Pionke and Wempner (1991), Levy (1991), Munz and Yang (1992), Kfouri and Wong (1993), Munz *et al.* (1993), Dreier *et al.* (1994), Munz and Yang (1994)).

The geometry near the free edge of the interface is described by the two contact angles θ_1 and θ_2 (see Fig. 1). Perfect bonding between the two materials is assumed. The stresses near the free edge of the interface can be obtained applying an Airy stress function and the boundary conditions at the free surfaces ($\theta = \theta_1$ and $\theta = \theta_2$) and the continuity of normal stress, shear stress and displacements at the interface between the two materials ($\theta = 0$). For details see Appendix.

For homogeneous, isotropic, elastic materials the stresses near the free edge of the interface in a joint with arbitrary geometry can be described by the relation

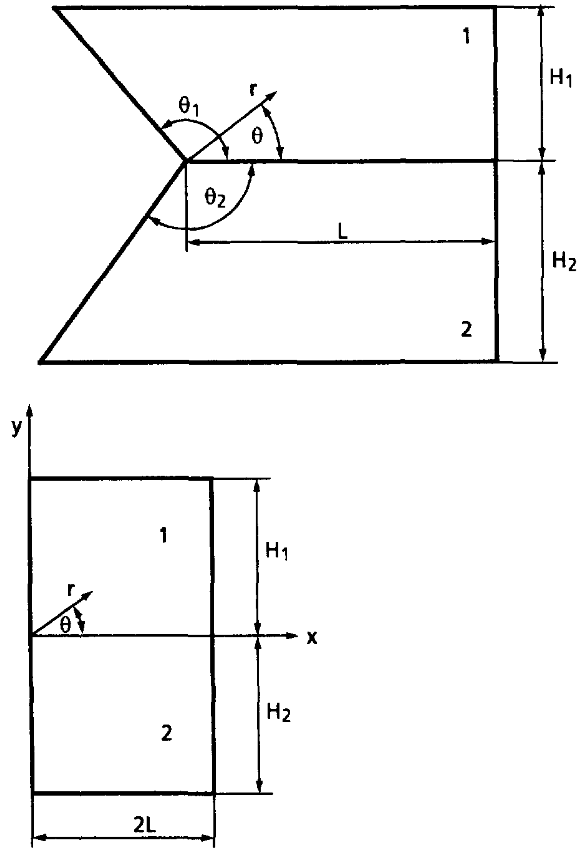


Fig. 1. The geometry and coordinates of bimaterial joints.

$$\sigma_{ij}(r,\theta) = \sum_{k=1}^{N} \frac{K_{Lk}}{(r/L)^{\omega_k}} f_{ijk}(\theta) + \sigma_o f_{ijo}(\theta) \tag{1}$$

where r, θ are polar coordinates (see Fig. 1), ω_k are stress exponents. L is a characteristic length of the joint, σ_o is the constant stress term, f_{ijk} are angular functions and K_{Lk} are stress intensity factors.

The regular stress term $\sigma_o f_{ijo}$ has to be introduced because of the thermal strains (see Suga *et al.* (1989)) and it is important, so close to the free edge of the interface. The stress exponents ω_k and the angular functions f_{ijk} depend on the elastic constants (E_1, ν_1, E_2, ν_2) and the contact angles θ_1 and θ_2 and are independent of the applied load or temperature. They can be calculated analytically. In many cases one or two stress exponents (ω_1 and ω_2) are positive and thus stress singularities exist. For specific combinations of material properties and contact angles complex eigenvalues of the problem exist (see Williams (1956), Hein and Erdogan (1971), Bogy and Wang (1971)). In those cases eqn (1) is no longer valid and the stress field has to be described by a different relation (see Yang and Munz (1995)). These cases are not considered here.

For a homogeneous temperature change in the joint the stress intensity factors K_{Lk} and the constant stress term σ_o can be written as

$$K_{Lk} = \Delta T(\alpha_1^* - \alpha_2^*) \cdot K_{Lk}^* \tag{2a}$$

$$\sigma_o = \Delta T(\alpha_1^* - \alpha_2^*) \cdot \sigma_o^* \tag{2b}$$

with

$$\alpha_i^* = \begin{cases} \alpha_i & \text{for plane stress} \\ \alpha_i(1+\nu_i) & \text{for plane strain.} \end{cases}$$

The quantities K_{Lk}^* and σ_o^* depend on the elastic constants and the contact angles. The K_{Lk}^*-values, in addition, depend on the overall geometry, e.g. H_1/L and H_2/L. The quantity σ_o^* can be calculated analytically (see Yang and Munz (1992), Munz *et al.* (1993), Yang and Munz (1994)), whereas the K_{Lk}^*-values, in general, have to be determined by numerical methods, e.g. the finite element method (FEM).

In this paper bonded quarter planes with $\theta_1 = -\theta_2 = 90^\circ$ are considered. For these joints only one singular term is necessary to describe the stress field near the singular point. The stress exponent can be calculated by solving a transcendent equation (see Suga *et al.* (1989)). The angular functions can be obtained analytically from the equations given by Munz and Yang (1994). The constant stress term is given for a homogeneous temperature change by

$$\sigma_o^* = \left[\frac{1}{E_1^*} - \frac{1}{E_2^*}\right]^{-1} \quad \text{with} \quad E_i^* = \begin{cases} \dfrac{E_i}{v_i} & \text{for plane stress} \\ \dfrac{E_i}{v_i(1+v_i)} & \text{for plane strain} \end{cases} \tag{3}$$

and $f_{yyo} = 1, f_{xxo} = f_{xyo} = 0$. From results obtained by applying the finite element method some empirical relations have been found for the stress intensity factor (see Munz and Yang (1992), Tilscher *et al.* (1994), Tilscher *et al.* (1995)). For a homogeneous change in temperature and $H_1/L \geqslant 2$, $H_2/L \geqslant 2$, $0.2 \leqslant \nu_1 \leqslant 0.4$ and $0.2 \leqslant \nu_2 \leqslant 0.4$ the stress intensity factor K_L can be related to the stress exponent and to the constant stress term σ_o by

$$-K_L/\sigma_o = 1-2.89\omega+11.4\omega^2-51.9\omega^3+135.7\omega^4-135.8\omega^5. \tag{4}$$

For other ratios of H_1/L and H_2/L relations for K_L are given by Tilscher *et al.* (1995). Therefore, it is now possible to calculate the stresses near the free edge of the interface for any bonded quarter planes with a homogeneous change in temperature without any further finite element calculations.

To the authors knowledge there are no studies on the analytical description of the stresses near the singular point in a dissimilar materials joint under an inhomogeneous change in temperature. In this paper a relation between the temperature distribution which changes only in the direction of the coordinate y and the stress intensity factor and the constant stress term is presented.

2. DETERMINATION OF THE STRESS INTENSITY FACTOR K_L AND THE CONSTANT STRESS TERM σ_O

The stress intensity factor and the constant stress term can be obtained from the results of a finite element calculation. An example of a finite element mesh is shown in Fig. 2 for the geometry $H_1 = H_2 = H$ and $H/L = 0.8$. Eight node elements were used in the calculations. The mesh had 3273 nodes and 1040 elements. The FE-code ABAQUS has been used. All calculations were for plane strain. The stress intensity factor K_L and the constant stress σ_o can be calculated from the stress distribution near the free edge of the interface by curve fitting. A least square procedure is applied, where the square of the difference between the stresses obtained by the finite element calculation σ_{ij}^{FE} and the stresses from the analytical expression given in eqn (1) is minimized

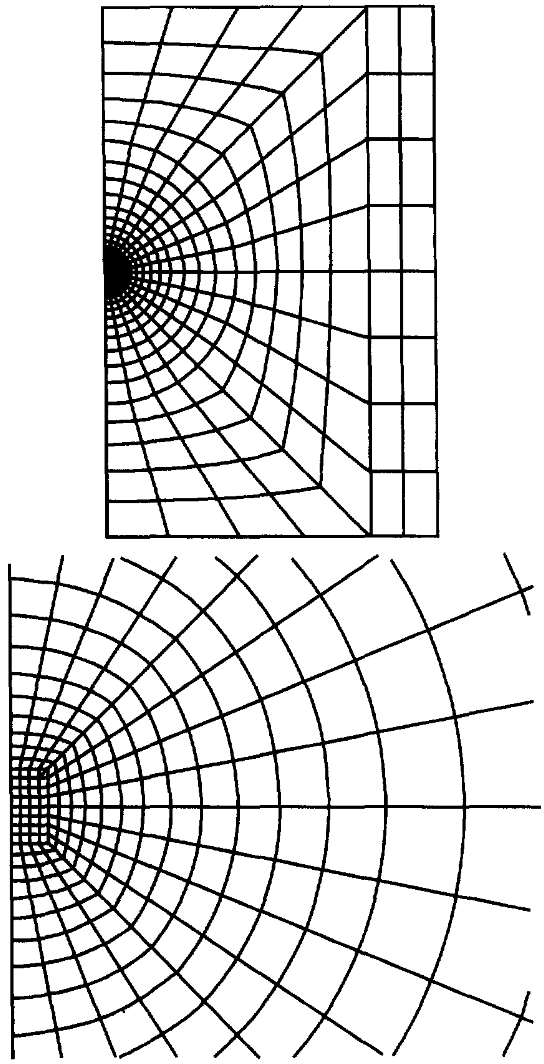


Fig. 2. The used finite element mesh.

$$\Pi = \sum_{l=1}^{M} \left\{ \sigma_{ij}^{FE}(r_l, \theta_l) - \sigma_o f_{ijo}(\theta_l) - \frac{K_L}{(r_l/L)^{\omega}} f_{ij}(\theta_l) \right\}^2 \tag{5}$$

where $ij = xx, yy, xy$. Π is minimized with respect to K_L and σ_o (for details see Munz and Yang (1993)). M is the number of points used to determine K_L and σ_o. The values of K and σ_o should be independent of the used stress component (σ_{xx}, σ_{yy} or σ_{xy}) and the angle θ_l. Our calculations showed that the determined values of K_L and σ_o from eqn (5) for different stress components and for different angles have differences less than 1%. In the following calculations the stress component σ_y for $\theta = 0$ was used. As an example Fig. 3 shows the effect of M on K_L and σ_o. The material data are those given in Section 3. In these calculations the lower limit of r was fixed and the upper limit r_{max} corresponding to M was increased. It can be seen that the obtained values of K_L and σ_o are constant. Deviations occur outside the range of application of eqn (1).

The constant stress term can be also determined analytically for inhomogeneous temperature distribution. This is shown in the Appendix. In all evaluated cases the agreement between the values of σ_o from the finite element calculations and the analytical equation (A10) in the Appendix was excellent.

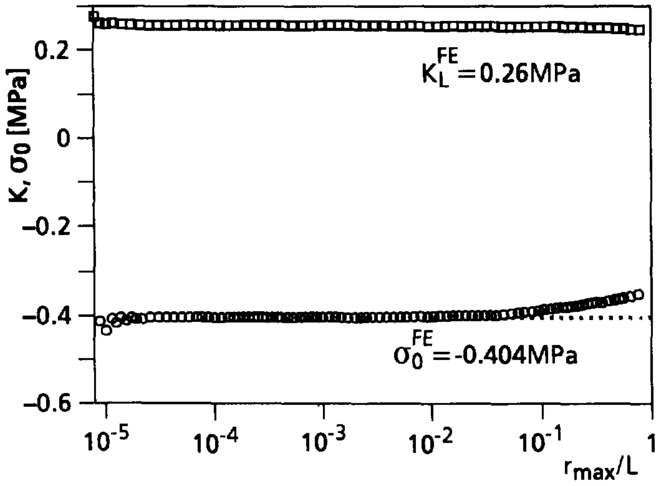


Fig. 3. Curves of K_L^{FE} and σ_o^{FE} vs r_{max}/L.

3. STRESS INTENSITY FACTORS FOR POLYNOMIAL TEMPERATURE DISTRIBUTION

For given thermal boundary conditions the temperature distribution in a component can be calculated in some cases analytically. In many cases, for complex thermal boundary conditions or complex geometries numerical methods have to be applied. Here it is assumed that the temperature distribution is known. From the given temperature distribution the thermal stresses can be calculated applying the finite element method. In principle, it is possible to calculate the stress intensity factor for each stress distribution according to each temperature distribution by applying eqn (5). It is, however, in many cases much more convenient to obtain the stress intensity factor from the temperature distribution without any further finite element or least square fitting procedure. For this purpose it is convenient to describe the temperature distribution by a polynomial expression

$$T_j(y/H) = T_o + \sum_{i=0}^{N} m_{ij}(y/H_j)^i \tag{6}$$

where T_o is the homogeneous temperature at the stress free state. The subscript j stands for material 1 ($j = 1$, $y > 0$) and material 2 ($j = 2$, $y < 0$). The change of temperature ($\Delta T = T - T_o$) at the interface is $\Delta T_o = m_{o1}$ in material 1 and $\Delta T_o = m_{o2}$ in material 2. For $m_{o1} = m_{o2}$ there is a continuous transition in temperature at the interface. The coefficients m_{ij} have to be obtained by fitting the given temperature distribution in a polynomial expression.

If the temperature distribution is described by a linear superposition of different terms, as given by eqn (6), in linear elasticity the stresses and the stress intensity factors can be obtained by a linear superposition of the corresponding results. Therefore, calculations were performed for temperature distributions which are $\Delta T = 0$ in one material and a power law in the other, as shown in Fig. 4. These temperature distributions are described only by the coefficients m_{ij}. For example, the temperature distribution characterised by m_{31} is

$$\begin{aligned} T &= T_o + m_{31}(y/H_1)^3 \quad \text{for } y \geq 0 \\ T &= T_o \quad \text{for } y < 0. \end{aligned} \tag{7}$$

The stress intensity factor K_{Lij} corresponding to each term in eqn (6)—$m_{ij}(y/H_j)^i$—is proportional to m_{ij}. The factor of proportionality is denoted C_{Lij}, i.e. $K_{Lij} = C_{Lij}m_{ij}$. For the temperature distribution given by eqn (6) the stress intensity factor can be then calculated by

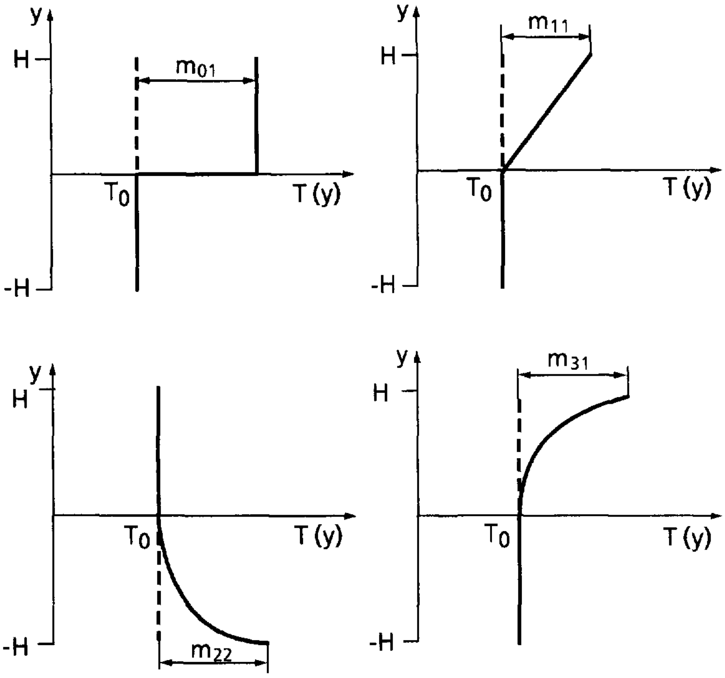


Fig. 4. Example of temperature distributions for determination of coefficients C_{Lij}.

$$K_L = \sum_{j=1}^{2} \sum_{i=0}^{N} C_{Lij} m_{ij}. \tag{8a}$$

For $m_{o1} = m_{o2}$ (continuous temperature transition at the interface) there is:

$$K_L = K_{Lh} \Delta T_o + \sum_{j=1}^{2} \sum_{i=1}^{N} C_{Lij} m_{ij} \tag{8b}$$

where K_{Lh} is the stress intensity factor for a homogeneous change of the temperature by one Kelvin.

The factors C_{Lij} depend on the overall geometry (H_1/L and H_2/L) and on the material properties. They have to be determined from finite element calculations applying eqn (5).

As a first example a joint with $H_1 = H_2 = H$ and $H/L = 0.8$ is considered. The material properties are $E_1 = 26.7$ GPa, $\nu_1 = 0.26$, $\alpha_1 = 2.08*10^{-6}$ K^{-1}, $E_2 = 300$ GPa, $\nu_2 = 0.32$, $\alpha_2 = 5.3*10^{-6}$ K^{-1}. Values of C_{Lij} for this example are given in Table 1. With these coefficients the stress intensity factor in this joint can be derived without any further finite

Table 1. Coefficients C_{Lij}

	C_{Lij} [MPa/K]	
ij	$H/L = 0.8$	$H/L = 2$
01	$-9.10 \cdot 10^{-2}$	$-9.83 \cdot 10^{-2}$
02	$3.51 \cdot 10^{-1}$	$3.62 \cdot 10^{-1}$
11	$-2.75 \cdot 10^{-2}$	$-1.34 \cdot 10^{-2}$
12	$7.32 \cdot 10^{-2}$	$3.58 \cdot 10^{-2}$
21	$-1.60 \cdot 10^{-2}$	$-4.47 \cdot 10^{-3}$
22	$-7.72 \cdot 10^{-2}$	$-2.74 \cdot 10^{-2}$
31	$-1.16 \cdot 10^{-2}$	$-2.05 \cdot 10^{-3}$
32	$6.90 \cdot 10^{-2}$	$1.98 \cdot 10^{-2}$
41	$-9.11 \cdot 10^{-3}$	$-1.10 \cdot 10^{-3}$
42	$-6.09 \cdot 10^{-2}$	$-1.52 \cdot 10^{-2}$
51	$-7.54 \cdot 10^{-3}$	$-6.46 \cdot 10^{-4}$
52	$5.42 \cdot 10^{-2}$	$1.22 \cdot 10^{-2}$

element calculation for any temperature distribution described by eqn (6). The coefficients C_{Lij} have an effect like a weight factor.

4. THE EFFECT OF THE GEOMETRY (H/L) ON THE STRESS INTENSITY FACTOR

For a homogeneous change in temperature Munz and Yang (1992) have found that for $H/L \geqslant 2$ the stress intensity factor K_L is constant, i.e. K_L reaches a limit value $K_{L\infty}$. If in eqn (1) the distance r is related to the height H instead of to L, then K_L has to be replaced by K_H. Between K_L and K_H the following relation holds:

$$K_H = K_L(L/H)^{\omega}. \tag{9}$$

Tilscher *et al.* (1995) have found that for a homogeneous change in temperature the stress intensity factor K_H reaches also a limit value $K_{H\infty}$ for $L/H \geqslant 2$. It was investigated if, for inhomogeneous temperature distribution, a similar tendency exists. We still consider joints with $H_1 = H_2 = H$. To describe the effect of the ratio H/L on the stress intensity factor it is useful to define new coefficients

$$C^*_{Lij} = C_{Lij}(H/L)^i. \tag{10}$$

In Fig. 5 the values of C^*_{Lij} are plotted vs H/L in a log-log-representation for $i = 0, 1, 2, 3, 4, 5$. It can be seen that for $H/L \geqslant 2$ a constant value $C^*_{Lij\infty}$ is obtained (only C^*_{L41} has a maximum for $H/L = 2$; this may be caused by numerical inaccuracies). For small H/L a straight line with the slope, $\omega + i$ exists. It is now favourable to use the other definition of the stress intensity factor K_H for small H/L (or large L/H). Using eqns (9) and (10), eqn (8a) can be rewritten as

$$K_H = \sum_{j=1}^{2}\sum_{i=0}^{N} C_{Lij}(L/H)^{\omega} m_{ij} = \sum_{j=1}^{2}\sum_{i=0}^{N} C^*_{Lij}(L/H)^{\omega+i} m_{ij} \tag{11a}$$

or

$$K_H = \sum_{j=1}^{2}\sum_{i=0}^{N} C_{Hij} m_{ij}. \tag{11b}$$

Between the coefficients C_{ij} the following relation applies:

$$C_{Hij} = C_{Lij}(L/H)^{\omega} = C^*_{Lij}(L/H)^{\omega+i}. \tag{12}$$

In Fig. 6 the same results as in Fig. 5c are plotted as $\log C_{Hij}$ vs $\log L/H$. For $L/H \geqslant 2$ a constant value $C_{Hij\infty}$ is obtained and for $L/H \leqslant 0.5$ a straight line with the slope $\omega + i$ exists. Therefore, different coefficients C_{ij} can be obtained for different ratios of H/L:

for $H/L \geqslant 2$ (or $L/H \leqslant 0.5$):

$$C^*_{Lij} = C^*_{Lij\infty}$$

$$C_{Lij} = C^*_{Lij\infty}(L/H)^i \quad \text{(from eqn (10))}$$

$$C_{Hij} = C^*_{Lij\infty}(L/H)^{\omega+i} \quad \text{(from eqn (12))}$$

for $H/L \leqslant 0.5$ (or $L/H \geqslant 2$):

$$C_{Hij} = C_{Hij\infty}$$

$$C_{Lij} = C_{Hij\infty}(H/L)^{\omega} \quad \text{(from eqn(12))}$$

$$C^*_{Lij} = C_{Hij\infty}(H/L)^{\omega+i} \quad \text{(from eqn (12))}.$$

As a conclusion of this section it can be stated that the stress intensity factors for a given material combination can be calculated by applying eqn (8) for any temperature distribution given by eqn (6). The coefficients C_{ij} obtained for $H/L = 2$ and $H/L = 0.5$ can be applied for any geometry with $H/L \geqslant 2$ and $H/L \leqslant 0.5$.

5. THE EFFECT OF MATERIAL PROPERTIES ON THE STRESS INTENSITY FACTOR

For a given temperature distribution and a given geometry the thermal stresses depend on the thermal expansion coefficients and on the elastic constants E_i and ν_i of the two materials. The effect of elastic constants on the stress exponent ω and on the angular functions f_{ij} is well known and independent of the temperature distribution. The effect of the material parameters on the constant stress term σ_o is given by eqn (A10).

The effect of the material properties on the stress intensity factor is very complicated. Only some first results are presented. From the definition of the coefficients C^*_{Li1} it is known that they correspond to a temperature change in material 1 and no temperature change in material 2. Therefore, they have to be independent of the thermal expansion coefficient in material 2. On the other hand the C^*_{Li1} are proportional to α^*_1. The coefficients C^*_{Li2} correspond to a temperature change in material 2 and no temperature change in material

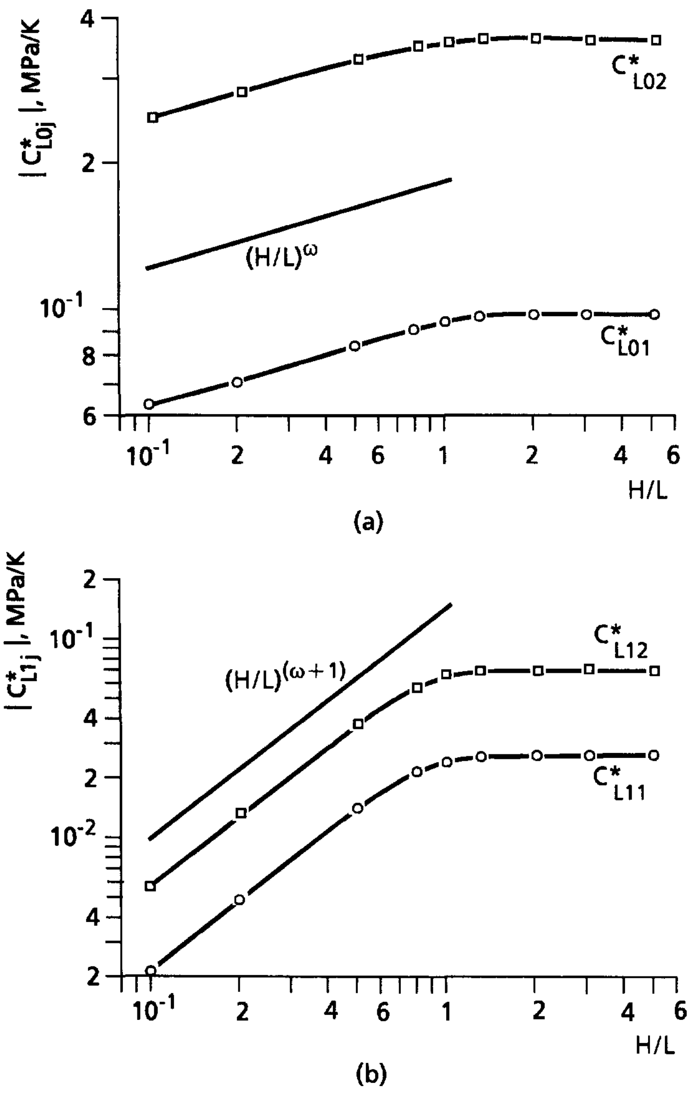


Fig. 5. Curve of log $|C^*_{Lij}|$ vs log(H/L). (*Continued opposite and overleaf.*)

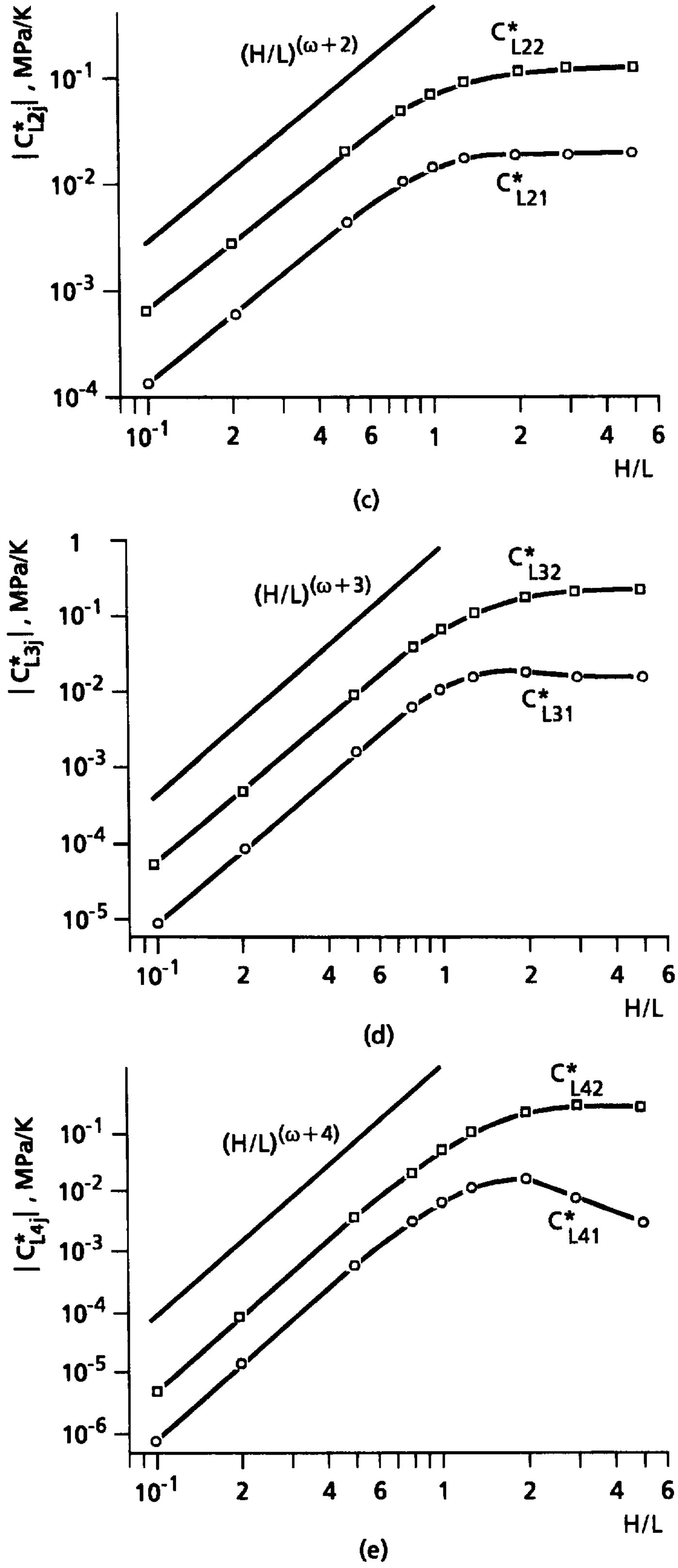
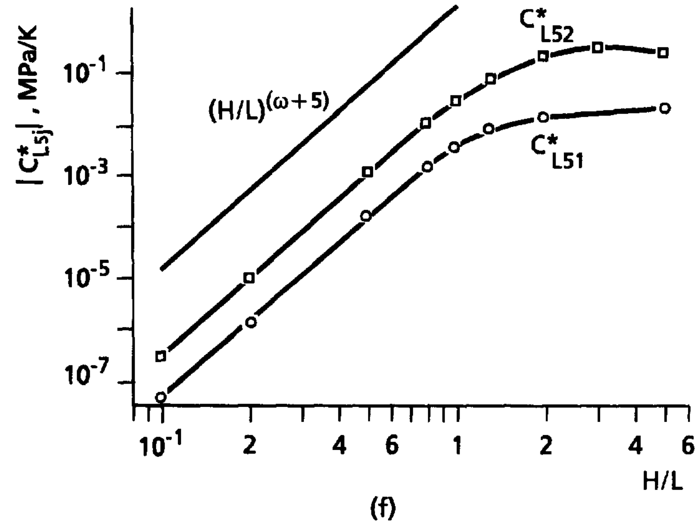


Fig. 5—(*Continued.*)



(f)

Fig. 5—(*Continued.*)

1 and they are thus independent of α_1 and proportional to α_2^*. Therefore, new coefficients $D^*_{Lij} = C^*_{Lij}/\alpha_j^*$ can be used, which are independent of the thermal expansion coefficients.

The effect of the elastic parameters on the stress intensity factor was not investigated systematically. As one example $D^*_{L11\infty}/(E_1+E_2)$ is plotted in Fig. 7 vs the relative difference in the Young's moduli $(E_1-E_2)/(E_1+E_2)$ for $\nu_1 = 0.2$ and $\nu_2 = 0.3$. Further studies are necessary to obtain a general overview of the effect of the elastic constants on the stress intensity factors.

6. EXAMPLE FOR AN ARBITRARY TEMPERATURE DISTRIBUTION

As an example a relative arbitrary temperature distribution according to

$$\Delta T = 300 + 500\sin(\pi y/H) + 10\exp(5y/H) \tag{13}$$

was chosen and firstly was applied to a geometry with $H/L = 0.8$ (see Fig. 8). The material data are the same as in Section 3. This temperature distribution was fitted by a 5th order

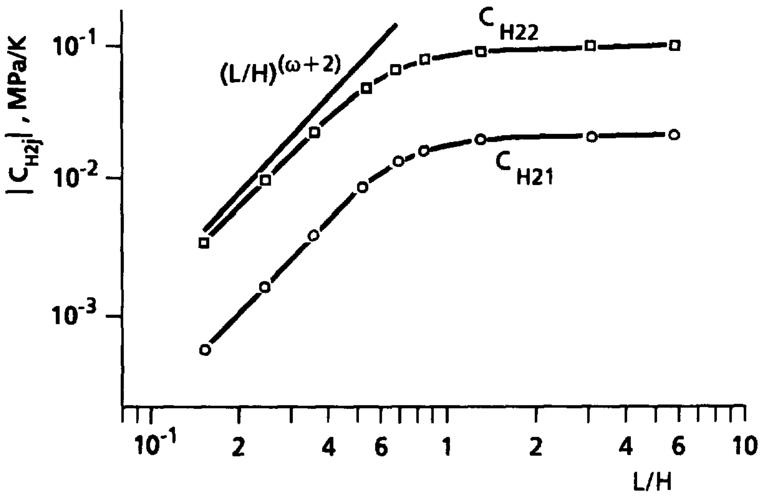


Fig. 6. Curves of $\log|C_{H2j}|$ vs $\log(L/H)$.

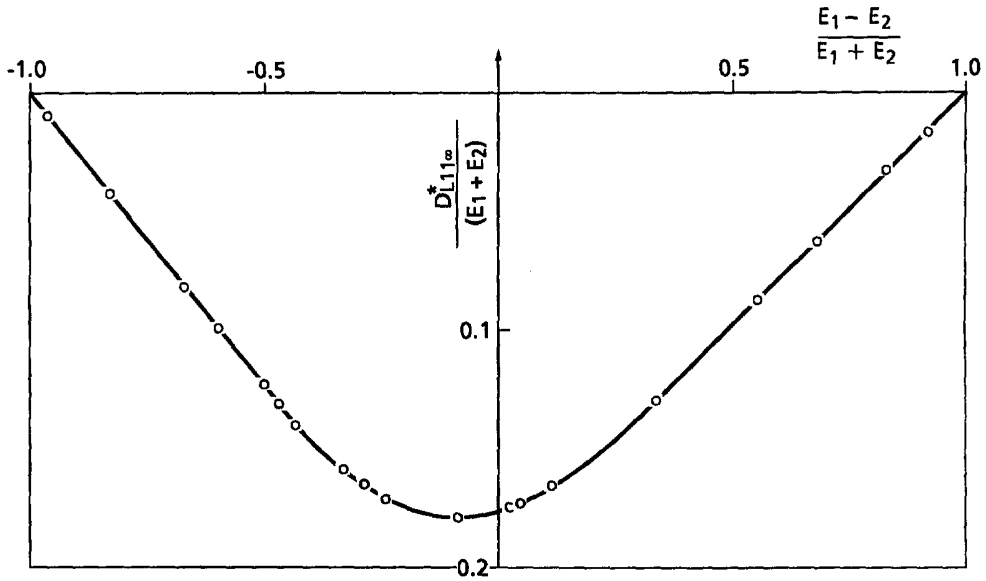


Fig. 7. Curve of $D^*_{L11\infty}/(E_1+E_2)$ vs $(E_1-E_2)/(E_1+E_2)$.

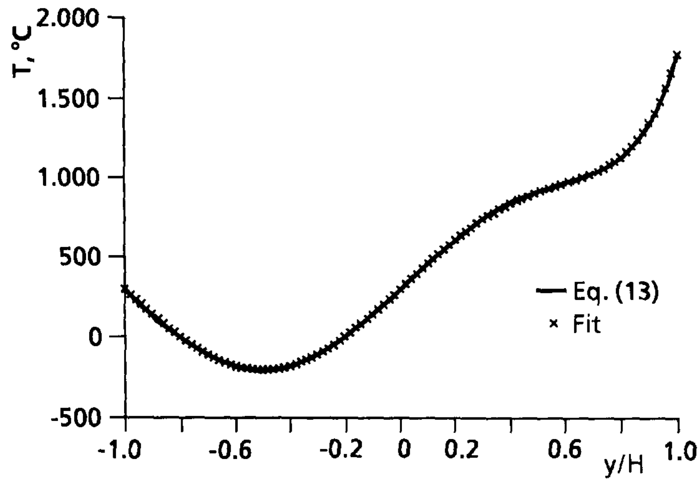


Fig. 8. Temperature distribution according to eqn (13).

polynomial expression through the points $y/H = 0$, ± 0.2, ± 0.4, ± 0.6, ± 0.8, ± 1.0. The coefficients m_{ij} are listed in Table 2. The analytical value of the stress intensity factor from eqn (8) with the coefficients from Table 1 is $K_L = 40.28$ MPa. A value of $K_L = 40.27$ MPa

Table 2. Coefficients m_{ij} (in Kelvin) for temperature distribution according to eqn (13)

i	$j = 1$	$j = 2$
0	310	310
1	1685	1597
2	−567.7	−148.3
3	180.2	−3473
4	−3724	−1711
5	3900	26.82

was obtained by direct determination from a finite element calculation. This shows that for an arbitrary inhomogeneous temperature distribution in the y-direction the stress intensity factor can be obtained accurately from eqn (8).

As a second example the geometry with $H/L = 2$ is considered. The material properties are the same as before. The calculated coefficients C_{Lij} are given in Table 1. For the temperature distribution given in eqn (13) a value of $K_L = 81.8$ MPa was obtained from eqn (8) and $K_L = 81.75$ from the FE-calculations.

In Fig. 9 the stress components from the finite element calculations are compared with those from the analytical expression given in eqn (1) by using K_L from eqn (8) for $H/L = 2$. It can be seen that the stresses near the singular point ($r/L < 0.01$) are described very accurately by the analytical expression.

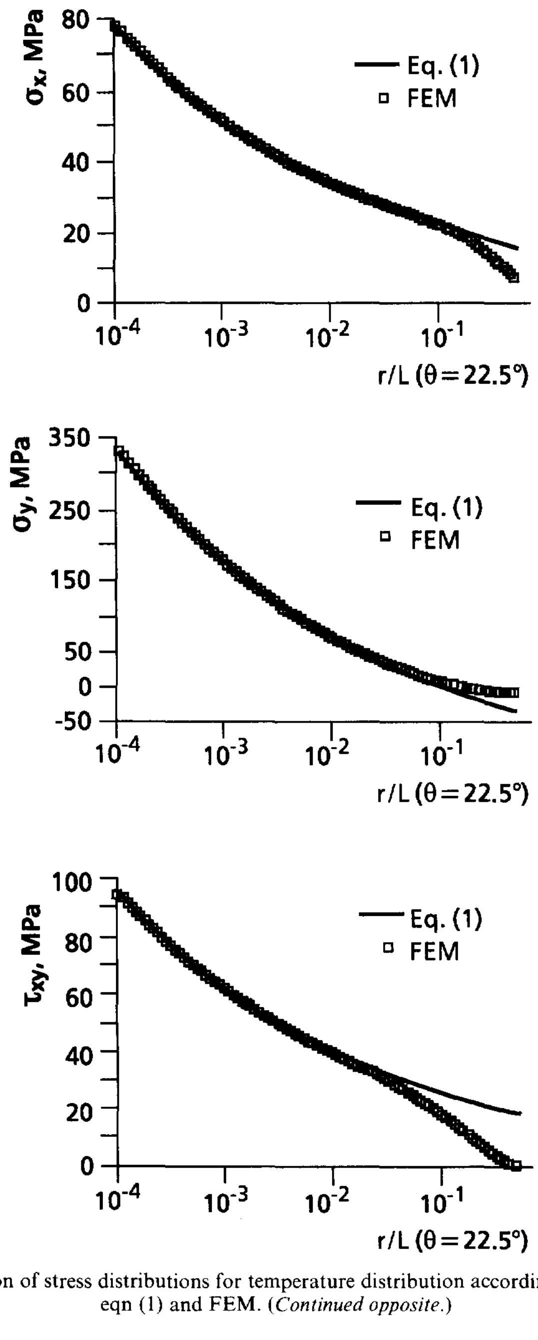


Fig. 9. Comparison of stress distributions for temperature distribution according to eqn (13) from eqn (1) and FEM. (*Continued opposite.*)

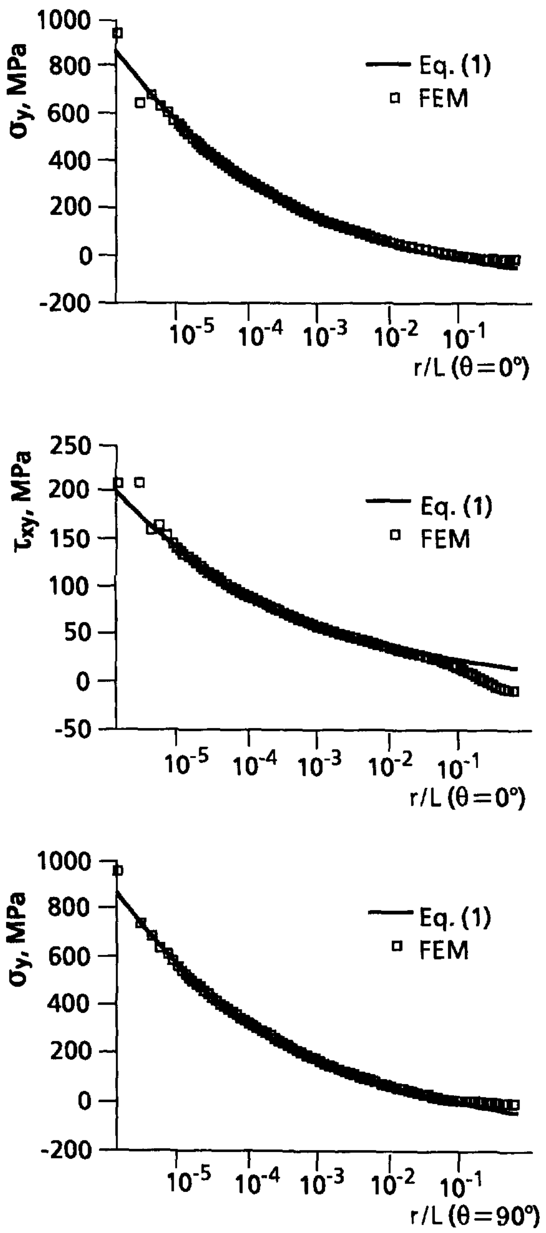


Fig. 9—(*Continued.*)

7. CONCLUSION

In bonded quarter planes with an inhomogeneous temperature distribution in a direction perpendicular to the interface the stress distribution near the free edge of the interface can be described by the sum of a singular term and a term, which is independent of the distance from the singular point. All parameters in these terms, with exception of the stress intensity factor of the singular term, can be calculated analytically.

For a polynomial temperature distribution an expression is presented for calculating the stress intensity factor. If for a given material combination and a given geometry the coefficients in this expression are known from some finite element calculations, the stress intensity factor and therefore, the stresses near the singular point can be calculated without any further finite element calculation for arbitrary temperature distributions. That means, the advantage of such a procedure is that for a given geometry and material combination only a few finite element calculations are necessary to obtain the coefficients C_{Lij}. Then for any thermal stress problems, e.g. thermal shock or thermal fatigue, where the variation of the singular stresses with time has to be known, the stresses can be calculated directly from the temperature distribution.

Some information about the effect of the height to length ratio of the joint on the stress intensity factor is given. There exist size-independent values of the coefficients for $H/L \geq 2$ and for $H/L \leq 0.5$.

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APPENDIX

Calculation of the regular stress term

The calculation follows the procedure described by Munz *et al.* (1993). We start from a stress function for two materials ($j = 1, 2$):

$$\phi_j(r,\theta) = r^{2-\omega}\{A_j\sin(\omega\theta)+B_j\cos(\omega\theta)+C_j\sin[(2-\omega)\theta]+D_j\cos[(2-\omega)\theta]\}$$
$$+r^2[A_{oj}\theta+B_{oj}+C_{oj}\sin(2\theta)+D_{oj}\cos(2\theta)]. \quad \text{(A1)}$$

The first part leads to the singular term and is not pursued further.

The stresses are obtained from:

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \quad \text{(A2a)}$$

$$\sigma_\theta = \frac{\partial^2\phi}{\partial r^2} \quad \text{(A2b)}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right). \tag{A2c}$$

For plane stress the relation between stresses and strains for thermal loading are:

$$\varepsilon_r = \frac{1}{E}(\sigma_r - v\sigma_\theta) + a\Delta T \tag{A3a}$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - v\sigma_r) + a\Delta T \tag{A3b}$$

$$\gamma_{r\theta} = \frac{1}{G}\sigma_{r\theta} \tag{A3c}$$

where G is the shear modulus.

The components of the strain are related to the displacements by

$$\varepsilon_r = \frac{\partial u}{\partial r} \tag{A4a}$$

$$\varepsilon_\theta = \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{A4b}$$

$$\gamma_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}. \tag{A4c}$$

Eight boundary conditions exist

$$\begin{aligned} &\theta = \theta_1 = 90^\circ: && \sigma_{\theta 1} = 0 && \sigma_{r\theta 1} = 0 \\ &\theta = \theta_2 = -90^\circ: && \sigma_{\theta 2} = 0 && \sigma_{r\theta 2} = 0 \\ &\theta = 0: && \sigma_{\theta 1} = \sigma_{\theta 2}, && \sigma_{r\theta 1} = \sigma_{r\theta 2} \\ & && u_1 = u_2, && v_1 = v_2. \end{aligned} \tag{A5}$$

From these conditions the following relations can be deduced

$$\begin{gathered} \frac{\pi}{2}A_{01} + B_{01} - D_{01} = 0 \\ -\frac{\pi}{2}A_{02} + B_{02} - D_{02} = 0 \\ A_{01} - 2C_{01} = 0 \\ A_{02} - 2C_{02} = 0 \\ B_{01} + D_{01} - B_{02} - D_{02} = 0 \\ A_{01} + 2C_{01} - A_{02} - 2C_{02} = 0 \\ 2\mu[B_{01}(1-v_1) - D_{01}(1+v_1)] - 2[B_{02}(1-v_2) - D_{02}(1+v_2)] = E_2(m_{02}a_2 - m_{01}a_1) \\ A_{01}\mu = A_{02} \end{gathered} \tag{A6}$$

with $\mu = E_2/E_1$.

The solution of these equations is

$$A_{01} = A_{02} = C_{01} = C_{02} = 0 \tag{A7a}$$

$$B_{01} = D_{01} = B_{02} = D_{02} = \frac{m_{02}a_2 - m_{01}a_1}{4\left[\frac{v_2}{E_2} - \frac{v_1}{E_1}\right]} = Z. \tag{A7b}$$

The regular stress terms are

$$\sigma_{r_o} = 2Z[1 - \cos(2\theta)] \tag{A8a}$$

$$\sigma_{\theta_o} = 2Z[1 + \cos(2\theta)] \tag{A8b}$$

$$\sigma_{r\theta_o} = 2Z\sin(2\theta) \tag{A8c}$$

or in cartesian coordinates

$$\sigma_{yo} = 4Z$$

$$\sigma_{xo} = \sigma_{xyo} = 0. \tag{A9}$$

Thus the quantity σ_o in eqn (1) can be written as

$$\sigma_o = 4Z = (m_{01}a_1 - m_{02}a_2)\cdot\Delta E \tag{A10}$$

with

$$\Delta E = \left[\frac{v_1}{E_1} - \frac{v_2}{E_2}\right]^{-1}. \tag{A11}$$

For plane strain a, v and E have to be replaced by

$$\alpha \to \alpha(1+v)$$

$$v \to \frac{v}{1-v}$$

$$E \to \frac{E}{1-v^2} \tag{A12}$$

leading to

$$\Delta E = \left[\frac{v_1(1+v_1)}{E_1} - \frac{v_2(1+v_2)}{E_2}\right]^{-1}. \tag{A13}$$